

**REPORT No. 78**

# **THE LIMITING VELOCITY IN FALLING FROM A GREAT HEIGHT**



**NATIONAL ADVISORY COMMITTEE  
FOR AERONAUTICS**



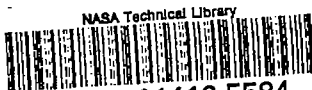
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# **THE LIMITING VELOCITY IN FALLING FROM A GREAT HEIGHT.**

**BY EDWIN BIDWELL WILSON**



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### THE LIMITING VELOCITY IN FALLING FROM A GREAT HEIGHT.

By EDWIN BIDWELL WILSON.

1. The fundamental characteristic of the vertical motion under gravity in a resisting medium is the approach to a final or limiting velocity  $U$ , whether the initial downward velocity is less or greater than  $U$ . The equations of motion are easily set up and integrated when the density of the medium is constant and the resistance varies as the square of the velocity.

A fact not so much stressed is the slowness of the approach to the limiting velocity  $U$ . For example the simple relation  $v^2 = 2gh$ , which neglects altogether the resistance of the atmosphere, shows that the height  $h$  of release from rest must be about 10,000 feet before the object will attain a velocity of as much as 800 ft./sec. even without the opposition of the air resistance; and when allowance is made for that resistance the height must be greater. If, therefore, terminal velocities of 900 or more feet per second, such as are customary with airplane bombs, are under consideration, it is only in the case of fall from a great height (upward of 10,000 feet) that the terminal velocity can be anywhere nearly approached.

Here, however, another difficulty enters. The resistance of the air varies with the density of the medium and this variation should not be assumed a priori to be negligible in the case of fall from heights of upward of 2 miles. In the standard table of densities at different levels, the ratio  $\rho/\rho_0$  of the density at altitude  $h$  to that at the earth has become about 0.74 at 10,000 feet, 0.62 at 15,000, and 0.44 at 25,000. The question thus arises as to whether the changing resistance of the air may be taken into account in some satisfactorily simple way for the discussion of vertical fall.

It is the purpose of this report to give that simple treatment of the problem which I have in the past offered to my classes at the Massachusetts Institute of Technology; there are undoubtedly other solutions, perhaps equally simple or simpler, but I have seen none.

2. The equation of motion is

$$W \frac{dv}{dt} = -Wg + c\rho v^2$$

if  $W$  be the <sup>weight</sup> mass,  $v$  the velocity upward, and  $c\rho v^2$  be the resisting force in pounds. With the substitution  $u = v^2$  the equation reduces to  $\frac{du}{dh} - \frac{2cg\rho}{W} u = -2g$  and is linear, but with a variable

coefficient  $\rho$ . The general solution may be indicated in the form

$$ue^{-\frac{2cg}{W} \int \rho dh} = -2g \int e^{-\frac{2cg}{W} \int \rho dh} dh + C$$

In order to perform the integration it is necessary (unless graphical methods are used) to have an expression for  $\rho$  as a function of  $h$  that is sufficiently accurate and at the same time of such form as to make integrable the two expressions

$$\int \rho dh \quad \text{and} \quad \int e^{-\frac{2cg}{W} \int \rho dh} dh + C$$

Some work will be saved if the first expression can itself be represented by an empirical equation directly without having first to determine  $\rho$  and then integrate that result.

3. Physically  $\int_0^h \rho dh$  is the amount of air between the earth and the height  $h$  in a column of unit cross section. This amount can fortunately be read off immediately from a table of barometric pressures  $p$  at different levels—in fact  $\int_0^h \rho dh$  is simply  $p_0 - p_1$ , the drop in pressure in pounds per square foot.

If  $F(h)$  is the amount of air in inches of mercury, the following table gives the value of  $F$  from  $h=0$  to  $h=24,000$  feet.

$h=0$	4,000	8,000	12,000	16,000	20,000	24,000
$F=0$	4.07	7.68	10.85	13.62	16.02	18.09

Now, as the exponential of  $F$  (multiplied by a constant) must be integrated, it is advisable to have a logarithmic expression to give the empirical relation between  $h$  and  $F$ . Try:

$$F(h) = b \log_e (1 + ah).$$

This holds for  $h=0$  and has two disposable constants  $a$  and  $b$ . A least-squares solution could be made to determine the best values for  $a$  and  $b$ ; but a sufficiently good result may be had by passing the curve through or near two sets of values in the table:

$$F_1 = b \log (1 + a h_1), \quad F_2 = b \log (1 + a h_2):$$

$$F_1 : F_2 = \log (1 + a h_1) : \log (1 + a h_2), \quad 1 + a h_1 = (1 + a h_2)^{F_1/F_2}$$

Let  $h_2 = 8,000$ . Then  $F_2 = 7.68$ . Let  $F_1 = 2F_2 = 15.36$ . Then  $h_1$  is just under 20,000 and may be taken as  $h_1 = 19,000$  by interpolation. Hence

$$1 + 19,000 a = (1 + 8,000 a)^2 = 1 + 16,000 a + 64,000,000 a^2$$

and

$$a = 3/64,000. \quad \text{Whence } b = 24.1.$$

Thus

$$F(h) = 24.1 \log_e (1 + 3h/64,000) = 55.5 \log_{10} (1 + 3h/64,000).$$

The values determined by this function  $F$  are

$h=0$	4,000	8,000	12,000	16,000	20,000	24,000
$F=0$	4.05	7.68	10.8	13.5	15.9	18.1

These check with the given values to less than 1 per cent, and therefore seem quite good enough.

4. The empirical formula just obtained must be transformed over to suitable units for insertion in the integral of the differential equation. Let  $U$  be the terminal or limiting velocity of the projectile in air of the density  $\rho_0$  at the earth's surface. Then

$$cg\rho_0 U^2 = Wg \text{ or } c = W/U^2 \rho_0$$

$$\frac{-2cg}{W} \int_0^h \rho dh = \frac{-2g}{\rho_0 U^2} \int_0^h \rho dh.$$

If the units be the pound and foot

$$\int_0^h \rho dh = \frac{13.6}{12} \times 62.5 \times F(h) = 1706 \log_e (1 + 3h/64,000).$$

Taking  $g = 32.17$  and  $\rho_0 = .07608$ , the result is

$$\frac{-2g}{\rho_0 U^2} \int_0^h \rho dh = \frac{-1,430,000}{U^2} \log \left( 1 + \frac{3h}{64,000} \right).$$

With an error of less than 1 per cent this may be written as

$$\frac{-2g}{\rho_0 U^2} \int_0^h \rho dh = - \left( \frac{1200}{U} \right)^2 \log \left( 1 + \frac{3h}{64,000} \right).$$

An error of 1 per cent seems quite within the range of accuracy possible in such work as this, where the variation of atmospheric conditions from the so-called standard table may at any particular time be considerable.

The integral is therefore

$$v^2 \left(1 + \frac{3h}{64,000}\right)^{-\left(\frac{1200}{U}\right)^2} = -2g \int \left(1 + \frac{3h}{64,000}\right)^{-\left(\frac{1200}{U}\right)^2} dh + C,$$

where the only parameter remaining is the terminal velocity  $U$  in air of standard density.

If in particular a particle starts from rest at the altitude  $H$ ,

$$v^2 \left(1 + \frac{3h}{64,000}\right)^{-\left(\frac{1200}{U}\right)^2} = 2g \int_h^H \left(1 + \frac{3h}{64,000}\right)^{-\left(\frac{1200}{U}\right)^2} dh.$$

The integral may be evaluated readily, as the integrand is a simple negative power, but leads to a complicated literal formula.

A simple case would be where  $U=1,200$ . Then

$$v^2 = 2g \left(1 + \frac{3h}{64,000}\right) \frac{64,000}{3} \log_e \left[ \frac{1 + \frac{3H}{64,000}}{1 + \frac{3h}{64,000}} \right]$$

This is the only case in which a logarithm enters; the others are algebraic, e. g., if  $U=600$ ,

$$v^2 = 2g \left(1 + \frac{3h}{64,000}\right)^4 \frac{64,000}{3 \times 3} \left[ \left(1 + \frac{3h}{64,000}\right)^{-3} - \left(1 + \frac{3H}{64,000}\right)^{-3} \right]$$

5. The formulas thus obtained should not be applied to calculate velocities which are large compared with that of sound. It is generally admitted that the simple square law of resistance does not hold for velocities much in excess of 800 ft./sec. Just how well the law holds below that figure may still be considered doubtful; but two recent authorities (E. Vallier, *Balistique Exterieur*, Encyclopedie des Sciences Mathematiques, tome IV, Vol. 6, fasc. 1, p. 15, Gauthier-Villars, Paris, 1913; and J. Prescott, London, *Phil. Mag.*, ser. 6, vol. xx, p. 332, Oct. 1917) seem to feel tolerably certain of the law for compact shell-like bodies up to 800 ft./sec. It was seen at the start that irrespective of air resistance, such a velocity would not be obtained from rest in a fall of 10,000 feet. If  $U=1,200$ , the velocity of 800 ft./sec. will be reached at the earth's surface only when the fall is through some 15,000 feet. (Whether high velocities might not be attained at intermediate levels is a question that should not be overlooked.)

6. It is often stated, as is indeed obvious, that a body falling in a medium of increasing density such as the air may reach a maximum velocity and be subsequently retarded before striking the earth. The terminal velocity toward which the body strives is of course greater at higher levels. The question of reaching a maximum velocity is therefore a question of balance between the height of fall and the natural resistance of the body relative to its weight. A body of high terminal velocity (in standard air) must fall from a very great height in order to attain a maximum greater than the speed with which it reaches the earth, whereas a body of low terminal velocity need not fall so far.

The maximum velocity of a body falling from rest is obtained by differentiating

$$v^2 = 2g \left(1 + \frac{3h}{64,000}\right)^{\left(\frac{1200}{U}\right)^2} \int_h^H \left(1 + \frac{3h}{64,000}\right)^{-\left(\frac{1200}{U}\right)^2} dh.$$

The derivative of an integral with respect to the lower limit is the negative of the integrand.



Hence for the maximum of  $v$ , or of  $v^2$

$$-1 + \left(\frac{1,200}{U}\right)^2 \left(1 + \frac{3h}{64,000}\right) \left(\frac{1200}{U}\right)^2 - 1 \left[ \frac{\left(1 + \frac{3H}{64,000}\right)^{-\left(\frac{1200}{U}\right)^2+1} - \left(1 + \frac{3h}{64,000}\right)^{-\left(\frac{1200}{U}\right)^2+1}}{-\left(\frac{1,200}{U}\right)^2+1} \right] = 0$$

For a given  $H$  this equation will determine the level  $h$  where the maximum velocity is attained.

If in particular this maximum should be at  $h=0$ , the altitude  $H$  of fall must be

$$H = \frac{64,000}{3} \left[ \left(\frac{1,200}{U}\right)^{\frac{2}{\left(\frac{1200}{U}\right)^2-1}} - 1 \right]$$

For example, if  $U=600$ ,  $H=12,500$ . The maximum velocity will be reached before striking the earth only if  $H > 12,500$ . If  $U=1,200$ , the bracket has the value  $e-1=1.73$ , and the maximum value will not be attained unless the drop is from over 35,000 feet—levels to which the empirical formulas used do not remain valid.

In any case of resisted fall in which a maximum of velocity is attained the value of that maximum velocity must be the value of the terminal velocity at that level, because the body has been gaining velocity in the rarer air above and will be losing velocity in the denser air below.

If, therefore, the velocity at  $h=0$  is below the terminal velocity  $U$ , it may safely be assumed that the value for  $h=0$  has not been exceeded; but if the value for  $h=0$  is larger than  $U$ , there has been a still larger velocity at some point of the path. The caution parenthetically suggested at the end of the last article can therefore be observed easily.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, September 6, 1919.

